Math 1020 Week 8

Series

Given a sequence $\{a_k\}$. Define

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

$$S_\infty = \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots$$

$\sum_{k=1}^{\infty} a_k$ is called a <u>series</u>

$\sum_{k=1}^{n} a_k$ is called a <u>partial sum</u>.

If $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists,

then $\sum_{k=1}^{\infty} a_k$ is said to be <u>convergent</u>

Otherwise $\sum_{k=1}^{\infty} a_k$ is said to be <u>divergent</u>

<u>eg1</u> $a_k = \frac{1}{k^2}$

Then $S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2^2} = \frac{5}{4}$

$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2} = \frac{49}{36}$

n	1	2	3	10	100	1000
S_n (4 dp)	1	1.25	1.3611	1.5498	1.6350	1.6439

<u>Fact</u> $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2}$ exists

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}}$$

eg 2 $a_k = \frac{1}{k}$ Then $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

square

n	1	2	10	10^3	10^6	10^{12}
S_n (4 dp)	1	1.5	2.9290	5.1874	9.7876	18.9980

almost double (increases logarithmically $\ln x^2 = 2\ln x$)

Fact $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \infty$ (DNE)

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty \text{ (DNE)}}$$

Why ∞?

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{>\frac{1}{2}} + \underbrace{\frac{1}{17} + \cdots + \frac{1}{32}}_{>\frac{1}{2}} + \cdots$$

$\Rightarrow$ greater than $\frac{1}{2} \cdot n$ for any integer n

Geometric Series

Consider sequence $\overset{a_1}{\overset{\|}{a}}, \overset{a_2}{\overset{\|}{ar}}, \overset{a_3}{\overset{\|}{ar^2}}, \overset{a_4}{\overset{\|}{ar^3}}, \cdots$

$\therefore\ a_k = ar^{k-1},\quad a_k = a_{k-1} r \quad (r = \text{common ratio})$

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots ar^{n-1}$$

$$S_\infty = \sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

Formula For $r \neq 1$,

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r} \leftarrow$$

a = first term

r = common ratio

n = number of terms

$$\sum_{k=1}^{\infty} ar^{k-1} \begin{cases} = \dfrac{a}{1-r} & \text{if } |r| < 1 \\ = \pm\infty & \text{if } r \geq 1,\ a \neq 0 \\ \text{DNE} & \text{if } r \leq -1,\ a \neq 0 \end{cases}$$

Pf

$$\begin{aligned} (1-r)(1+r+r^2+\cdots+r^{n-1}) &= 1 + r + r^2 + \cdots + r^{n-1} \\ &\quad - r - r^2 - \cdots - r^{n-1} - r^n \\ &= 1 - r^n \end{aligned}$$

$$\begin{aligned} \therefore \sum_{k=1}^{n} ar^{k-1} &= a + ar + ar^2 + \cdots + ar^{n-1} \\ &= a(1 + r + r^2 + \cdots + r^{n-1}) \\ &= \frac{a(1-r^n)}{1-r} \quad \text{for } r \neq 1 \end{aligned}$$

Take $n \to \infty$, obtain result for $\sum_{k=1}^{\infty} ar^{k-1}$

Rmk Some variations

first term

$$\sum_{k=1}^{n} ar^k = ar + ar^2 + \cdots + ar^n = \frac{ar(1-r^n)}{1-r}$$

$$\sum_{k=0}^{n} ar^k = \underbrace{a + ar + \cdots + ar^n}_{n+1 \text{ terms}} = \frac{a(1-r^{n+1})}{1-r}$$

eg Compute the followings.

first term   common ratio   number of terms

① $$\sum_{k=1}^{10} \frac{2}{(-3)^k} = \frac{2}{-3}\cdot\frac{1-\left(\frac{1}{-3}\right)^{10}}{1-\left(\frac{1}{-3}\right)} = \frac{2}{-3}\cdot\frac{1-\frac{1}{3^{10}}}{\frac{4}{3}} = -\frac{1}{2}\left(1-\frac{1}{3^{10}}\right)$$

② $$\sum_{k=1}^{\infty} \frac{2}{(-3)^k} = \frac{2}{-3}\cdot\frac{1}{1-\left(\frac{1}{-3}\right)} = \frac{2}{-3}\cdot\frac{1}{\frac{4}{3}} = -\frac{1}{2}$$

③ $$\sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^{2k+1} = \frac{1}{5}\sum_{k=1}^{\infty}\left(\frac{1}{25}\right)^k = \frac{1}{5}\cdot\frac{\frac{1}{25}}{1-\frac{1}{25}} = \frac{1}{120}$$

④ $$\sum_{k=1}^{\infty} -\pi^k = -\infty \ \text{(divergent)} \qquad \left(\begin{array}{l} r=\pi\geq 1 \\ a=-\pi<0 \end{array}\right)$$

⑤ $$\sum_{k=1}^{n} (-1)^k = (-1)\frac{1-(-1)^n}{1-(-1)} = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

⑥ $\sum_{k=1}^{\infty} (-1)^k$ diverges

eg

$$\sum_{n=1}^{\infty} \frac{2^{3n+1}+(-1)^n}{3^{2n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{2^{3n+1}}{3^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{3^{2n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{2\cdot 8^n}{\frac{1}{3}\cdot 9^n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{3}\cdot 9^n}$$

$$= 6\sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n + 3\sum_{n=1}^{\infty}\left(-\frac{1}{9}\right)^n$$

$$= 6\cdot\frac{\frac{8}{9}}{1-\frac{8}{9}} + 3\cdot\frac{-\frac{1}{9}}{1-\left(-\frac{1}{9}\right)}$$

$$= 48 - \frac{3}{10}$$

$$= \frac{477}{10}$$

Telescoping Series

Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} \cdots$$

Trick : Partial fractions !

Sol

Let $\quad \dfrac{1}{k(k+1)} = \dfrac{A}{k} + \dfrac{B}{k+1}$

$$1 = (k+1)A + kB = k(A+B) + A$$

$$\Rightarrow \begin{cases} A+B=0 \\ A=1 \end{cases} \Rightarrow A=1,\ B=-1$$

$$\therefore \quad \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n-1} \right)$$

$$= 1 - \frac{1}{n+1}$$

Take $n \to \infty$,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right) = 1 - 0 = 1$$

Ex Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+4k+3}$

Ans : $\dfrac{5}{12}$

eg $\sum_{k=1}^{n} \ln\left(1+\frac{2}{k}\right)$

$= \sum_{k=1}^{n} \ln\left(\frac{k+2}{k}\right)$

$= \sum_{k=1}^{n} \left[\ln(k+2) - \ln k\right]$

$= (\cancel{\ln 3} - \ln 1) + (\cancel{\ln 4} - \ln 2)$

$+ (\cancel{\ln 5} - \cancel{\ln 3}) + (\cancel{\ln 6} - \cancel{\ln 4})$

$+ (\cancel{\ln 7} - \cancel{\ln 5}) + \ldots$

$+ \left[\ln(n+1) - \cancel{\ln(n-1)}\right] + \left[\ln(n+2) - \cancel{\ln(n)}\right]$

Only 4 terms left

$= -\ln 2 - \ln 1 + \ln(n+1) + \ln(n+2)$

$= \ln \frac{(n+1)(n+2)}{2}$

Rmk $\lim_{n\to\infty} \ln \frac{(n+1)(n+2)}{2} = \infty$ (DNE)

$\therefore \sum_{k=1}^{\infty} \ln\left(1+\frac{2}{k}\right)$ is divergent

eg Show $\sum_{k=1}^{45} \cos(2k-1)^\circ = \frac{\csc 1^\circ}{2}$

Sol Use formula

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$\sin 1^\circ \left[\sum_{k=1}^{45} \cos(2k-1)^\circ\right]$

$= \sum_{k=1}^{45} \cos(2k-1)^\circ \sin 1^\circ$

$= \frac{1}{2} \sum_{k=1}^{45} \left[\sin 2k^\circ - \sin(2k-2)^\circ\right]$

$= \frac{1}{2}\left[(\cancel{\sin 2^\circ} - \sin 0^\circ) + (\cancel{\sin 4^\circ} - \cancel{\sin 2^\circ})\right.$

$+ (\cancel{\sin 6^\circ} - \cancel{\sin 4^\circ}) + (\cancel{\sin 8^\circ} - \cancel{\sin 6^\circ})$

$\left. + \ldots + (\sin 90^\circ - \cancel{\sin 88^\circ})\right]$

$= \frac{1}{2}(\sin 90^\circ - \sin 0^\circ) = \frac{1}{2}$

$\Rightarrow \sum_{k=1}^{45} \cos(2k-1)^\circ = \frac{1}{2\sin 1^\circ} = \frac{\csc 1^\circ}{2}$

Tests for convergence of Series

A series may or may not converge:

$$\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e - 1 \quad \text{(convergent)}$$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots = \infty \quad \text{(divergent)}$$

Q Any test to determine if $\sum_{n=1}^{\infty} a_n$ is convergent?

n-th term Test

$$\text{If } \lim_{n\to\infty} a_n \begin{cases} \neq 0 \text{ or DNE, then } \sum_{n=1}^{\infty} a_n \text{ diverges} \\ = 0 \text{ then no conclusion} \end{cases}$$

limit of the sum

limit of the terms

Rmk The test is equivalent to say

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Rightarrow \lim_{n\to\infty} a_n = 0$$

Ratio Test

$$\text{If } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \begin{cases} < 1, \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges} \\ > 1 \text{ or } = \infty, \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges} \\ = 1 \text{ or DNE } (\neq \infty), \text{ then no conclusion} \end{cases}$$

Rmk n-th term Test <u>cannot</u> be used to show a series converges but Ratio Test can be.

eg Determine convergent / divergent?

① $\displaystyle\sum_{k=1}^{\infty} \cos\left(\frac{k^2-1}{2k^2+k}\right)$ $\quad \left(\because a_k = \cos\left(\frac{k^2-1}{2k^2+k}\right)\right)$

Sol

$$\lim_{k\to\infty} \cos\left(\frac{k^2-1}{2k^2+k}\right) = \lim_{k\to\infty} \cos\left(\frac{1-\frac{1}{k^2}}{2+\frac{1}{k}}\right)$$

$$= \cos\frac{1}{2} \neq 0$$

By "k-th term test", $\displaystyle\sum_{k=1}^{\infty} \cos\left(\frac{k^2-1}{2k^2+k}\right)$ diverges

② $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{3+(-1)^n}$ $\left(\therefore a_n = \frac{2+(-1)^n}{3+(-1)^n}\right)$

<u>Sol</u> $\displaystyle\frac{2+(-1)^n}{3+(-1)^n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{3}{4} & \text{if } n \text{ is even} \end{cases}$

$\therefore \displaystyle\lim_{n\to\infty} \frac{2+(-1)^n}{3+(-1)^n}$ DNE

By n-th term test, $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{3+(-1)^n}$ diverges

③ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $\left(\therefore a_n = \frac{2^n}{n!}\right)$

<u>Sol</u>

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$$

By Ratio test, $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n!}$ converges

④ $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ $\left(\therefore a_n = \frac{1+2^n}{1+3^n}\right)$

<u>Sol</u>

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1+2^{n+1}}{1+3^{n+1}}}{\frac{1+2^n}{1+3^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1+2^{n+1}}{1+2^n}\cdot\frac{1+3^n}{1+3^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{1}{2^n}+2}{\frac{1}{2^n}+1}\cdot\frac{\frac{1}{3^n}+1}{\frac{1}{3^n}+3}\right|$$

$$= \frac{2}{1}\cdot\frac{1}{3} = \frac{2}{3} < 1$$

By Ratio test, $\displaystyle\sum_{n=1}^{\infty}\frac{1+2^n}{1+3^n}$ converges

<u>Rmk</u> Usually, to show a series is

$\begin{cases} \text{divergent, try n-th term test} \\ \text{convergent, try ratio test} \end{cases}$